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THEORETICAL AND EMPIRICAL RESULTS ON THE ANALYTIC GRADUATION OF FERTILITY CURVES

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1. INTRODUCTION

1 A. The diagram of age-specific fertility rates for a population, based on data for a calendar period, say, will typically picture a curve which looks much like a left-skewed unimodal probability density, such as a gamma density or some of the beta densities (starting just below age 15), for instance, but with superposed fluctuations. Unless the population is very large, the diagram of the sequence of fertility rates, plotted against age, will have quite a ragged appearance. It is frequently assumed that "real fertility" would be portrayed by a smooth curve and that the irregularities of the observed curves are due to accidental circumstances. The observed fertility rates are then regarded as "raw" or primary estimates of the underlying "real" rates, and graduation is employed to get a smoother curve.

1 B. When the smooth curve is produced by fitting a nice, parametric function to the original data, we call it analytic graduation. The present paper reports some findings on the following question : Given that one has chosen a particular graduating function with a specified parametrization, which method should one select to fit the function to the observed fertility data?

Some previous research (Hoem, 1972) has shown that a modified minimum *chi*-square method cannot be outdone by any other weighted or unweighted least squares method or by any moment method, in the sense that asymptotically as the population size increases the modified minimum *chi*-square estimators have minimal variances within the class of estimators considered. This holds for the estimators of the parameters of the fitting function as well as for the estimators of the function values (i.e., of the "true" fertility rates at all ages). So far, one has not known, however, to what extent this theoretical result has practical consequences in that the numerical values of the asymptotic variances of *chi*-square estimators are noticeably smaller than the corresponding variance values for reasonable competitors like least squares estimators.

1 C. We have applied analytic graduation methods to a number of empirical fertility curves, and this paper reports briefly on some selected typical findings, It turns out that estimated asymptotic variances (and the corresponding estimated coefficients of variation) almost without exception are much smaller for rates graduated by least squares than for the original ungraduated rates. In every case which we have investigated, there is also some further real gain in estimated variance in going from least squares graduation to minimum chi-square graduation. On a rare occasion, the extra gain in variance is as "low" as some 10 per cent, but it frequently is higher, and it can be substantial. Thus the optimality (in terms of estimated asymptotic variances) of the modified minimum chi-square method does have considerable practical interest.

1 D. The presentation goes as follows :

In Section 2 we summarize results given in the previous paper, but restated here in a form meant to be more easily accessible. The account is largely phrased in terms of fertility graduation, and it is of course relevant to this problem, but the reader should bear in mind that the theory is by no means limited to the case of fertility. It can be applied to any type of occurrence/exposure rate.

Section 3 then contains the selected empirical results. They are based on curves of observed fertility rates, and they are relevant for such curves as well as for other curves of vital rates of a similar form, such as marriage rates, rates of migration, etc.

2. THEORETICAL RESULTS

2 A. The "raw" age-specific fertility rates in a set will have been calculated for a number of age groups, which for simplicity we shall take to be the single-year age intervals $\alpha, \alpha+1, \cdots, \beta$. Let $\hat{\boldsymbol{\lambda}} = (\hat{\lambda}_\alpha, \hat{\lambda}_{\alpha+1}, \cdots, \hat{\lambda}_\beta)'$ denote the vector of "raw" rates, the prime signifying a transpose. These rates are to be graduated by means of some sequence of parametric functions

$$\mathbf{g}(\boldsymbol{\theta}) = (g_\alpha(\boldsymbol{\theta})\, g_{\alpha+1}(\boldsymbol{\theta}), \cdots, g_\beta(\boldsymbol{\theta}))',$$

that is, one is required to select some value $\hat{\boldsymbol{\theta}}$ of the vector $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_r)$, such that $\mathbf{g}(\hat{\boldsymbol{\theta}})$ fits as well as possible to the "raw" rates in $\hat{\boldsymbol{\lambda}}$. In our particular empirical situation, $g_x(\boldsymbol{\theta})$ is the value of, say, the Hadwiger function at argument x, so that

$$g_x(\boldsymbol{\theta}) = \frac{RH}{T\sqrt{\pi}} \left(\frac{T}{x-D}\right)^{3/2} \exp\left\{-H^2\left(\frac{T}{x-D} + \frac{x-D}{T} - 2\right)\right\}.$$

Other investigations (Hoem and Berge, 1974) have shown that it pays to take

$$\theta_1 = R,\ \theta_2 = D + T\left\{\left(1 + \frac{16}{9}H^4\right)^{1/2} - 1\right\} \Big/ \left(\frac{4}{3}H^2\right),$$

$$\boldsymbol{\theta}_3 = T + D, \text{and } \theta_4 = \frac{1}{2}\ T^2/H^2$$

as the basic parameters of the graduation. (If we regard $g_x(\boldsymbol{\theta})$ as a function of a continuous x, $g_x(\boldsymbol{\theta})/R$ then becomes a probability density with mode θ_2, mean θ_3, and variance θ_4.)

We shall take this to mean that there is some underlying "true" vector $\boldsymbol{\lambda}^0 = \{\lambda^0_\alpha, \ldots, \lambda^0_\beta\}'$ of fertility rates for the age intervals in question, and that the task at hand is to remove as well as possible the random deviation $\hat{\boldsymbol{\lambda}} - \boldsymbol{\lambda}^0$. For our purposes, $\boldsymbol{\lambda}^0$ can be represented with sufficient accuracy by $\mathbf{g}(\boldsymbol{\theta}^0)$, where $\boldsymbol{\theta}^0$ is a corresponding "true" value of $\boldsymbol{\theta}$. Then, $\boldsymbol{\theta}^0$ is estimated by $\hat{\boldsymbol{\theta}}$ and $\boldsymbol{\lambda}^0$ by $\mathbf{g}(\hat{\boldsymbol{\theta}})$.

2 B. The fitting can be done in various ways, but the ordinary (i.e., unweighted) least squares method and the modified minimum chi-square method are immediate and prominent candidates. They consist, of course, of minimizing

$$Q_{LS}(\boldsymbol{\theta}) = N \sum_{x=\alpha}^{\beta} \{\hat{\lambda}_x - g_x(\boldsymbol{\theta})\}^2$$

and

$$Q_{CS}(\boldsymbol{\theta}) = \sum_{x=\alpha}^{\beta} \frac{\{\hat{\lambda}_x - g_x(\boldsymbol{\theta})\}^2}{\hat{\sigma}^2_x/N} = \sum_{x=\alpha}^{\beta} \frac{\{B_x - L_x g_x(\boldsymbol{\theta})\}^2}{B_x}$$

Here, N denotes the total number of women under observation, i.e., the number of women who are potential contributors to the numbers of liveborn babies counted. B_x is the number of liveborn babies with mothers at age x at the time of childbearing, and L_x is the total number of person-years observed at age x, so that $\hat{\lambda}_x = B_x/L_x$. Furthermore, $\hat{\sigma}^2_x = N\hat{\lambda}_x/L_x$. We can regard $\hat{\sigma}^2_x/N$ as an estimator of the asymptotic variance of $\hat{\lambda}_x$. These two methods are members of a whole class of procedures which are generated in the following way:

Let $\mathbf{M}$ be a square matrix of elements m_{xy} $(x, y = \alpha, \alpha+1, \ldots, \beta)$ which may (but need not) be random variables (by depending on the data). Let

$$Q_{\mathbf{M}}(\hat{\boldsymbol{\theta}}) = N\{\hat{\boldsymbol{\lambda}} - \mathbf{g}(\boldsymbol{\theta})\}' \,\mathbf{M}\,\{\hat{\boldsymbol{\lambda}} - \mathbf{g}(\boldsymbol{\theta})\} = N \sum_x \sum_y m_{xy}\{\hat{\lambda}_x - g_x(\boldsymbol{\theta})\}\,\{\hat{\lambda}_y - g_y(\boldsymbol{\theta})\},$$

and assume that there exists a value of $\boldsymbol{\theta}$, say $\hat{\boldsymbol{\theta}}(\mathbf{M})$, which minimizes $Q_{\mathbf{M}}(\boldsymbol{\theta})$. Then $\mathbf{g}\{\hat{\boldsymbol{\theta}}(\mathbf{M})\}$ represents one possible graduation of $\boldsymbol{\lambda}$.

If $\mathbf{M} = \mathbf{I}$, the graduation is by means of least squares.

Let $\hat{\boldsymbol{\sigma}} = \text{diag}\,(\hat{\sigma}^2_\alpha, \hat{\sigma}^2_{\alpha+1}, \ldots, \hat{\sigma}^2_\beta)$. Then, if $\mathbf{M} = \hat{\boldsymbol{\sigma}}^{-1} = \text{diag}\,(1/\hat{\sigma}^2_\alpha, \ldots, 1/\hat{\sigma}^2_\beta)$, $\mathbf{g}\{\hat{\boldsymbol{\theta}}(\mathbf{M})\}$ represents a graduating by modified minimum chi-square.

2 C. In order to choose between the various possible methods of fitting $\mathbf{g}(\boldsymbol{\theta})$ to $\hat{\boldsymbol{\lambda}}$, we must know something about their statistical properties. A framework for a discussion of these can be found in a previous paper (Hoem, 1972), and we shall recapitulate a few of the main results stated there. These results hold under assumptions which were given in that paper, and which we shall not repeat in full here, but which we can reasonably assume to hold for the kind of practical situation which we have in mind.

We shall assume, then, that as the population size N increases, $\hat{\boldsymbol{\lambda}}$ is asymptotically multinormally distributed with mean $\boldsymbol{\lambda}^0$ and some covariance matrix $\boldsymbol{\Sigma}_0/N$. [In our empirical studies of fertility, we take $\boldsymbol{\Sigma}_0$ to be diagonal, i.e., we take the $\hat{\lambda}_x$ to be asymptotically independent.]

We shall assume also that as $N \to \infty$, $\mathbf{M}$ converges in probability to some positive definite matrix $\mathbf{M}_0$.

Then $\hat{\boldsymbol{\theta}}(\mathbf{M})$ is asymptotically multinormally distributed with mean $\boldsymbol{\theta}^0$ and a covariance matrix $\boldsymbol{\Sigma}/N$, which satisfies

$$\boldsymbol{\Sigma} = \mathbf{A}\,\boldsymbol{\Sigma}_0\,\mathbf{A}', \tag{2.1}$$

with

$$\mathbf{A} = (\mathbf{J}_0'\,\mathbf{M}_0\,\mathbf{J}_0)^{-1}\,\mathbf{J}_0'\,\mathbf{M}_0. \tag{2.2}$$

Here, $\mathbf{J}_0 = \mathbf{J}(\boldsymbol{\theta}^0)$, where

$$\mathbf{J}(\boldsymbol{\theta}) = \begin{Bmatrix} \dfrac{\partial}{\partial\theta_1} g_\alpha(\boldsymbol{\theta}), \ldots, \dfrac{\partial}{\partial\theta_r} g_\alpha(\boldsymbol{\theta}) \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \dfrac{\partial}{\partial\theta_1} g_\beta(\boldsymbol{\theta}), \ldots, \dfrac{\partial}{\partial\theta_r} g_\beta(\boldsymbol{\theta}) \end{Bmatrix}.$$

Similarly, $\mathbf{g}(\hat{\boldsymbol{\theta}}(\mathbf{M}))$ is asymptotically multinormally distributed with mean $\boldsymbol{\lambda}^0 = \mathbf{g}(\boldsymbol{\theta}^0)$ and (singular) covariance matrix $\mathbf{J}_0\,\boldsymbol{\Sigma}\,\mathbf{J}_0'/N$. If, in particular, $\mathbf{M}_0 = \boldsymbol{\Sigma}_0^{-1}$ then $\boldsymbol{\Sigma}$ is equal to

$$\boldsymbol{\Sigma}_{0,0} = (\mathbf{J}_0'\,\boldsymbol{\Sigma}_0^{-1}\,\mathbf{J}_0)^{-1}.$$

This will be the case in our practical situation if we use the modified minimum *chi*-square method. In the general case, $\boldsymbol{\Sigma}$ will equal $\boldsymbol{\Sigma}_{0,0}$ if $\mathbf{M} = \hat{\boldsymbol{\Sigma}}_0^{-1}$, where $\hat{\boldsymbol{\Sigma}}_0$ is a consistent estimator of $\boldsymbol{\Sigma}_0$.

One can prove that for any M_0, $\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_{0,0}$ will be positive semidefinite, and so will $\mathbf{J}_0\,\boldsymbol{\Sigma}\,\mathbf{J}_0' - \mathbf{J}_0\,\boldsymbol{\Sigma}_{0,0}\,\mathbf{J}_0'$. This means, among other things, that the asymptotic variance of each $\hat{\theta}_i\,(\mathbf{M})$ will be minimized if $\mathbf{M}$ is chosen such that $\mathbf{M}_0 = \boldsymbol{\Sigma}_0^{-1}$. Similarly, the asymptotic variance of each $g_x(\hat{\boldsymbol{\theta}}(\mathbf{M}))$ will be minimized by the same choice of $\mathbf{M}$. Thus, in our particular empirical investigation, the use of *this* criterion of optimality will imply that one should use modified minimum *chi*-square graduation or some method with the same asymptotic properties, such as the maximum likelihood method described by Hoem (1972).

3. SOME EMPIRICAL RESULTS

3 A. We have applied the theory of the previous Section to a substantial number of Norwegian fertility curves, and a comprehensive presentation of our empirical results will be given in a forthcoming Working Paper. The present Section contains a brief account of two such cases for purposes of illustration.

3 B. We have selected the fertility curve for the city of Oslo, 1968—71, as well as the curve for the same period for an aggregate of 13 communes on the Norwegian West Coast, viz., Kvitsøy, Bokn, Utsira, Austevoll, Sund, Øygarden, Austrheim, Fedje, Solund, Askvoll, Selje, Sande, and Giske. The observed fertility curves have been plotted in Figures 1 and 2, respectively. The curve for Oslo represents a low level of fertility and a rather symmetrical age pattern of fertility for present-day Norway. (The total fertility rate by the *chi*-square fit is 1.9, while the corresponding modal and mean ages at childbearing are 25.2 and 27.3, respectively.) In contrast to this, the other curve represents a high fertility level for the Norway of today (TFR = 3.5) and a particularly skew age-pattern. (The modal and mean ages at childbearing by the chi-square fit are 24.1 and 28.4, respectively). These curves thus correspond to quite different patterns of fertility.

3 C. For each set of data, the "raw" fertility rates $\hat{\lambda}_x$ have been calculated for females for single-year age intervals, counting live offspring of both sexes. Age x corresponds to age as of December 31 in any observational year, and the contributions to L_x of any year is the arithmetic mean of the number of x-year olds at the beginning and end of the year, age being calculated as of Dec. 31 of *that* year.

The Hadwiger function of Subsection 2 A, with the parametrization $\theta = (\theta_1, \cdots, \theta_4)$ given there, has been fitted to the rates for ages 16 to 44. Then θ_1 corresponds to the total fertility rate, θ_2 to the modal age of childbearing, θ_3 to the corresponding mean age, and θ_4 to the variance of the age-pattern of fertility.

We have made graduations based on the method of least squares as well as separate graduations of the same data based on the modified minimum chi-square method. The graduated rates have been plotted in both Figures. Eyeball inspection judges the fit to be acceptable. As compared with the fit to other curves we have investigated, that of the two examples are about average.

The numerical calculation have been carried out on a Honeywell-Bull H6060 computer by means of a program developed by Berge (1974) on the basis of various algorithms previously published by others.

3 D. Let us denote the min.—χ^2 estimator for $\boldsymbol{\theta}$ by $\hat{\boldsymbol{\theta}}$, and the least squares estimator by $\boldsymbol{\theta}^*$. The corresponding estimates for the two sets of data mentioned have been listed in Table 1. In the case of the curve for Oslo, the estimates are pretty much the same for both methods. For the other curve, the two methods give somewhat different parameter estimates.

OBSERVED AND GRADUATED CURVES
Females, Oslo and 13 Norwegian communes 1968 — 71,

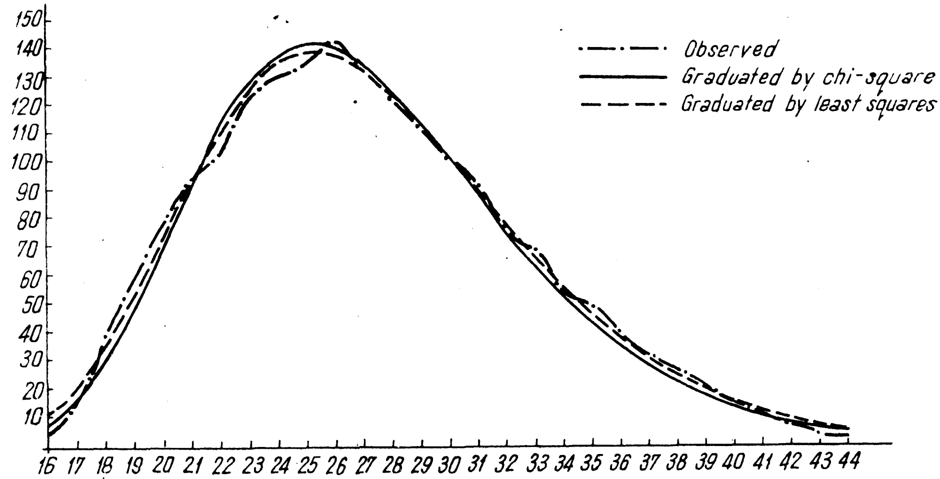

Fig. 1. Oslo Live offspring of both sexes

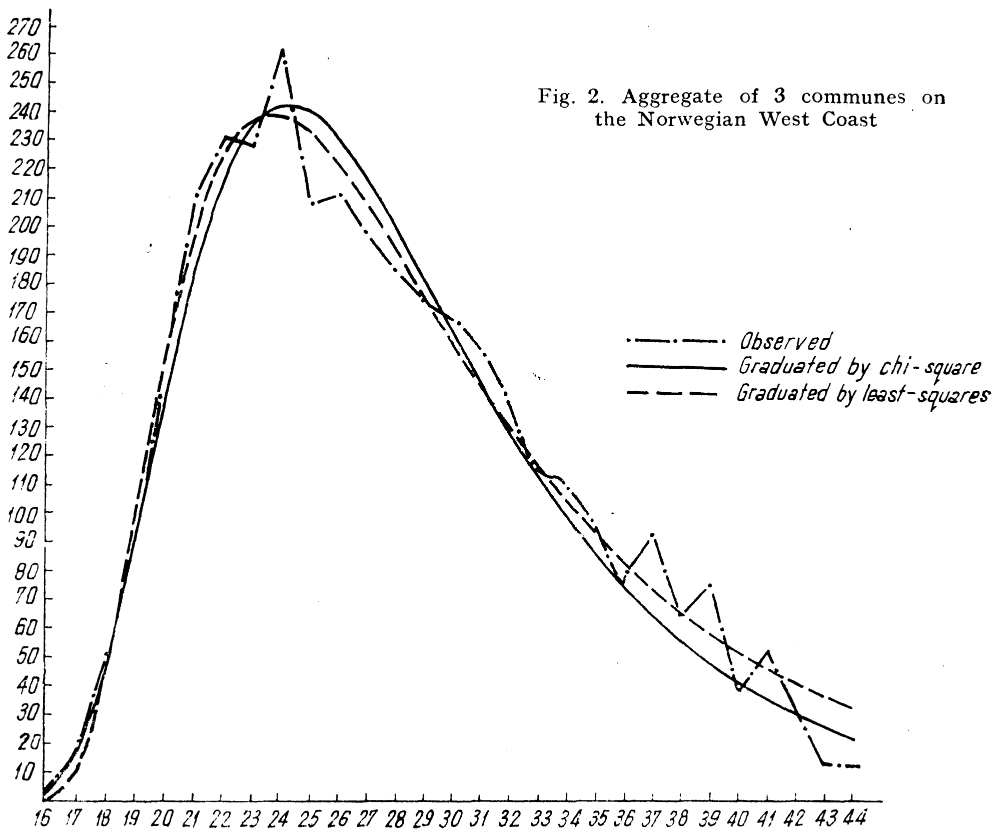

Fig. 2. Aggregate of 3 communes on the Norwegian West Coast

Table 1

Estimates of basic graduation parametres

Estimand	Min. statistic	Oslo, 1968–71		13 communes on the West Coast of Norway, 1968–71	
		By min. $-\chi^2$	By least squares	By min. $-\chi^2$	By least squares
θ_1 = R		1.88	1.93	3.50	3.72
θ_2 = mode		25.23	25.28	24.12	23.62
θ_3 = mean		27.28	27.40	28.44	29.70
θ_4 = variance		32.46	35.53	54.00	83.04
	χ^2 LSQ	186.1	$2.7 \cdot 10^{-4}$	41.9	$42.8 \cdot 10^{-4}$

Graduation of fertility curves by means of the Hadwiger function.
Single-year age intervals for ages 16–44. Rates per 1 female, counting live offspring of both sexes.

3 E. We now turn to variance estimation. We have proceeded as follows. Let $\hat{\mathbf{J}} = \mathbf{J}(\hat{\boldsymbol{\theta}})$, $\hat{\mathbf{A}} = \{\hat{\mathbf{J}}'(\hat{\boldsymbol{\sigma}}/N)^{-1}\hat{\mathbf{J}}\}^{-1}\,\hat{\mathbf{J}}'(\hat{\sigma}/N)^{-1}$, $\mathbf{A}^* = (\hat{\mathbf{J}}'\,\hat{\mathbf{J}})^{-1}\hat{\mathbf{J}}'$, $\hat{\Sigma}/N = \hat{\mathbf{A}}(\hat{\boldsymbol{\sigma}}/N)\hat{\mathbf{A}}'$, and $\boldsymbol{\Sigma}^*/N = \mathbf{A}^*\ (\hat{\boldsymbol{\sigma}}/N)\ (\hat{\mathbf{A}}^*)'$.
Then, $\hat{\boldsymbol{\Sigma}}/N$ is our estimator of the asymptotic covariance matrix of $\hat{\boldsymbol{\theta}}$, and $\boldsymbol{\Sigma}^*/N$ is our estimator of the corresponding matrix for $\boldsymbol{\theta}^*$, on the assumption that the true age-specific fertility rates are represented adequately by the Hadwiger curve. Similarly,

$$\hat{\mathbf{J}}(\hat{\boldsymbol{\Sigma}}/N)\ \hat{\mathbf{J}}' \text{ and } \hat{\mathbf{J}}\ (\boldsymbol{\Sigma}^*/N)\hat{\mathbf{J}}'$$

are our estimators of the asymptotic covariance matrices of $\mathbf{g}(\hat{\boldsymbol{\theta}})$ and $\mathbf{g}(\boldsymbol{\theta}^*)$, respectively.

Estimates of the asymptotic variances of the $\hat{\theta}_i$ and θ_i^* have been computed and listed in Table 2. The *chi*-square method seems impressively much better than the method of least squares for these estimands.

Table 2

Estimated asymptotic variances of estimators of basic graduation parameters[1)]

Parameter	Oslo, 1968–71		13 Norwegian communes, 1968–71	
	Min.$-\chi^2$ graduation[2)] (1)	Least squares graduation (NB: In per cent of (1).) (2)	Min. $-\chi^2$ graduation[2)] (3)	Least squares graduation (NB: In per cent of (3).) (4)
θ_1 = R	142	117	6 283	122
θ_2 = mode	3 609	132	44 281	166
θ_3 = mean	1 720	186	50 255	182
θ_4 = variance	201 572	216	17 999 297	201
ΣL^*	369 178.0		18 900.5	

1) Corresponding to min. $-\chi^2$ estimates of Table 1.
2) Multiplied by 10^6.

Corresponding variance estimates for graduated age-specific fertility rates for selected ages have been listed in Table 3, along with estimates for coefficients of variation. Column 4 shows that *chi*-square graduation of

Table 3

Estimated asymptotic variances and coefficients of variation[1]) of estimators of fertility rates for some selected ages

Age as of Dec. 31	Variances				Coefficients of variation			
	$\frac{\hat{\sigma}_x^2}{N} = \frac{\hat{\lambda}_x}{L_x}$ (1)[2]	Min.-χ^2 graduation (2)[2])	Least squares graduation (NB: In per cent of (2).) (3)[3])	(1) in per cent of (2) (4)[3])	$\frac{\hat{\sigma}_x/\sqrt{N}}{\hat{\lambda}_x}$ (5)[4])	Min.-χ^2 graduation (6)[4])	Least squares graduation (NB: In per cent of (6).) (7)[3])	(5) in per cent of (6) (8)[3])
			a. Oslo, 1968 – 71					
16	0.360	0.186	224	194	16.01	5.87	150	273
18	3.144	0.454	175	692	4.63	2.20	132	210
20	4.836	0.795	112	608	2.83	1.26	106	224
24	6.424	1.053	112	610	1.95	0.75	106	261
28	9.775	1.198	115	816	2.56	0.88	107	292
35	5.572	0.325	152	1713	4.77	1.33	123	358
40	1.425	0.117	225	1215	7.88	2.50	150	315
44	0.202	0.040	238	501	18.26	4.15	154	440
			b. 13 Norwegian communes, 1968 – 71					
16	2.206	0.839	390	263	57.75	42.76	197	135
18	51.202	13.548	231	378	15.08	7.59	152	199
20	219.575	45.390	119	484	9.49	4.65	109	204
24	393.465	44.963	117	875	7.58	2.77	108	273
28	317.285	35.990	116	882	9.71	3.03	107	320
35	173.826	9.776	138	1778	13.48	3.68	118	367
40	63.956	5.960	180	1073	21.32	5.99	134	356
44	18.703	3.836	203	488	35.36	8.93	141	396

1) Corresponding to the min. $-\chi^2$ estimates of Table 1.
2) Multiplied by 10^6, i.e., corresponding to rates per 1 000 females.
3) Calculated from figures with six effective digits.
4) Multiplied by 100.

the "raw" rates can result in a substantial reduction in variance, by a factor of 2 to 18 in the cases reported here. Column 3 shows that there is some real gain in using the *chi*-square method rather than least squares for these estimands as well. For the central ages (in the twenties), the gain is some ten to twenty per cent. In the tails of the curve, the *chi*-square method is at least twice as good as least squares, as judged by the variance estimates. Given the much larger weight placed on the tail ages by the former method than by the latter, such a pattern is not surprising, of course.

The variance reduction achieved by using least squares estimates rather than the "raw" rates can be gauged by dividing each of the entries in column 4 by the corresponding entry in column 3. For most ages, the variance is reduced by a factor of 2 to 12. For age 16 in each of the two cases reported, however, the estimated asymptotic variance of the least squares estimate of the fertility rate *exceeds* that of the "raw" rate (by 15 and 48 per cent for the two curves, respectively). Such "reversals" occur occasionally at isolated ages in our data sets.

Columns 7 and 8 give similar results for the coefficient of variation.

These numerical estimates show that for this type of fertility curves, the optimality of the modified minimum chi-square method is of practical importance and is not only of theoretical interest. This conclusion is substantiated by the rest of our experience with fertility curves.

3 F. The variance estimators introduced above are functions of the chi-square estimator $\hat{\boldsymbol{\theta}}$. We have computed similar variance estimates based on the least squares estimator $\boldsymbol{\theta}^*$. The numerical results do not seem to contain much beyond what we report here.

3 G. The modified minimum chi-square value for the curve of the 13 communes is 41.9, which corresponds to the upper 1.8 percentage point of the *chi*-square distribution with 25 $(=29-4)$ degrees of freedom. Most of the chi-square values which we have calculated for curves of regional populations in Norway are better than this. For the larger communities, the *chi*-square value is invariably big, however, and our curve for Oslo is a case in point. (See line 5 of Table 1). We take this as an indication of systematic deviations between the observed and the graduated curves. This corresponds to earlier findings by others in attempts at analytic graduation of *mortality* rates.

Sometimes, deviations between the observed and the graduated curve are regarded as caused at least in part by particular historical events of some substantive interest, or as an effect of cohort differences. In such a case, the graduated curve can still be useful in providing a "standard" with which the observed curve is compared to bring out systematic deviations more clearly. We regard this as an important aspect of fertility graduation (and a potentially more realistic one than the straightforward superposed random fluctuations interpretation), and we plan to discuss it in a future communication.

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